AN
INVESTIGATION
OF
THE PRINCIPLES
OF
PROGRESSIVE AND ROTATORY MOTION.

BY THE

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AN INVESTIGATION, &c.

THE communication of motion from impact is well known to conſtitute a conſiderable part of that branch of natural philoſophy called mechanics; and as all our enquiries therein are directed, either to aſſiſt us in thoſe operations which add to the conveniences of life, or to explain, for the ſatisfaction of the mind, thoſe changes which we daily ſee ariſe from the effects of bodies on each other, it might naturally have been expected that the attention of philoſophers would have been engaged, firſt in the inveſtigation of ſuch caſes as moſt frequently occur from the accidental action of one body on another, before they had proceeded to others leſs obvious. A little conſideration will convince any one how ſeldom it happens, in the colliſion of two bodies, that their centers of gravity and point of contact lie in the line of direction of the ſtriking body, yet few writers on mechanics have extended their enquiries any further than this ſimple caſe. It muſt however be acknowledged, that the action of bodies on each other, in directions *not* paſſing through their center of gravity, affords a ſubject at leaſt curious in ſpeculation; for my own part, I have little doubt but that it might be rendered extremely uſeful to the practical mechanic. I. BERNOULLI was the firſt who publiſhed any thing on this ſubject. He found the point about which a body at reſt would begin to revolve when ſtruck by another body, obſerving however that D. BERNOULLI had alſo diſcovered the ſame: he has alſo mentioned the curve deſcribed by that point in the progreſſive motion of the body, and has directed a method of enquiry by which the velocities of the bodies may be found after the ſtroke, which comprehends all he has done on the ſubject. Two years afterwards D. BERNOULLI publiſhed a paper on progreſſive and rotatory motion, containing nothing more than what I. BERNOULLI had before given us, and, what is a little extraordinary, ſays in the introduction, *de tali quidem percuſſione nihil adhuc, quantum ſcio, publici juris factum fuit ab iis, qui de motu corporum a percuſſione egerunt.* EULER has alſo inveſtigated the velocities of the bodies after impact in a manner ſomewhat different, but has rendered it much more intricate by a fluxional calculus. To any one, however, who attentively conſiders the ſubject, the theory muſt ſtill appear to be extremely imperfect, as, independent of principles not more ſelf-evident than the propoſitions they are intended to demonſtrate, which both I. and D. BERNOULLI have aſſumed in their inveſtigations, a great variety of other circumſtances equally intereſting naturally ariſe in an enquiry into this matter, [redacted] and which are absolutely neceſſary towards underſtanding the principles of the motion of the bodies after impact. This induced me to conſider the ſubject with ſome attention, and preſuming that I have not been altogether unſucceſsful in my endeavours to render the theory more perfect, I determined to lay the reſult of my enquiries before the Royal Society. I thought it expedient, for the ſake of perſpicuity, to divide the whole into diſtinct Propoſitions; and as the moſt ſimple caſes are beſt underſtood, I have firſt conſidered the caſe of the action of a body on a lever having a corpuſcle at each end: and I was the more induced to treat the ſubject in this manner, as moſt of the principles can be immediately applied to any number of corpuſcles, in conſequence of which the general inveſtigations are rendered more eaſy and ſatisfactory.

PROP. I.

Let A *and* B *be two indefinitely ſmall bodies connected by a lever void of gravity, and ſuppoſe a force to act at any point* D *perpendicularly to the lever, to find the point about which the bodies will begin to revolve.*

From the property of the lever, the effect of the force acting at D (fig. 1.) on the body A is to the effect on B as BD : AD; hence the ratio of the ſpaces A*m*, B*n*, deſcribed by the bodies A and B in the firſt inſtant of their motion, will be as $\frac{BD}{A} : \frac{AD}{B}$; join *mn*, and if neceſſary produce that line and AB to meet in C, which will manifeſtly be the point about which the bodies begin to revolve. Hence from ſimilar figures BC : AC : : $\frac{AD}{B}(\propto Bn) : \frac{BD}{A}(\propto Am) : : A \times AD : B \times BD$, or $DC - DB : AD + DC : : A \times AD : B \times BD$, and conſequently $DC = \frac{A \times AD^2 + B \times BD^2}{B \times BD - A \times AD}$, and therefore D is the center of percuſſion or oſcillation to the point of ſuſpenſion C.

Cor. 1. Hence, whatever be the magnitude of the ſtroke at D, the point C will remain the ſame.

Cor. 2. If the force acts at the center of gravity G, the bodies will have no circular motion, for in this caſe $B \times BD - A \times AD = 0$, and therefore DC becomes infinite.

Cor. 3. If the force acts at one of the bodies, the center of rotation C will coincide with the other body.

Cor. 4. If the lever had been in motion before the ſtroke, the point C, at the inſtant of the ſtroke, would not have been diſturbed.

PROP. II.

Let a given quantity of motion be communicated to the lever at D, *to determine the velocity of the center of gravity* G.

The ſpace $\mathrm{A}m$, deſcribed by the body A in the firſt inſtant of motion, is as $\frac{\mathrm{DB}}{\mathrm{A}}$: now $\mathrm{CG} = \mathrm{CD} - \mathrm{DG} = \mathrm{CD} - \mathrm{AG} + \mathrm{AD} = \frac{\mathrm{A} \times \mathrm{AD}^2 + \mathrm{B} \times \mathrm{BD}^2}{\mathrm{B} \times \mathrm{BD} - \mathrm{A} \times \mathrm{AD}} - \mathrm{AG} + \mathrm{AD} = \frac{\mathrm{B} \times \mathrm{BD} \times \mathrm{BG} + \mathrm{A} \times \mathrm{AD} \times \mathrm{AG}}{\mathrm{B} \times \mathrm{BD} - \mathrm{A} \times \mathrm{AD}}$; alſo $\mathrm{CA} = \mathrm{CD} + \mathrm{DA} = \frac{\mathrm{A} \times \mathrm{AD}^2 + \mathrm{B} \times \mathrm{BD}^2}{\mathrm{B} \times \mathrm{BD} - \mathrm{A} \times \mathrm{AD}} + \mathrm{DA} = \frac{\mathrm{B} \times \mathrm{BD} \times \mathrm{AB}}{\mathrm{B} \times \mathrm{BD} - \mathrm{A} \times \mathrm{AD}}$; hence we have $\frac{\mathrm{B} \times \mathrm{BD} \times \mathrm{AB}}{\mathrm{B} \times \mathrm{BD} - \mathrm{A} \times \mathrm{AD}} (\mathrm{AC}) : \frac{\mathrm{BD}}{\mathrm{A}} (\propto m\mathrm{A}) :: \frac{\mathrm{B} \times \mathrm{BD} \times \mathrm{GB} + \mathrm{A} \times \mathrm{AD} \times \mathrm{AG}}{\mathrm{B} \times \mathrm{BD} - \mathrm{A} \times \mathrm{AD}} (\mathrm{CG}) : \frac{\mathrm{B} \times \mathrm{BD} \times \mathrm{GB} + \mathrm{A} \times \mathrm{AD} \times \mathrm{AG}}{\mathrm{A} \times \mathrm{B} \times \mathrm{AB}} \propto \mathrm{G}\omega$ the velocity of the center of gravity: hence if the motion be communicated at G, the velocity becomes as $\frac{\mathrm{B} \times \mathrm{GB}^2 + \mathrm{A} \times \mathrm{AG}^2}{\mathrm{A} \times \mathrm{B} \times \mathrm{AB}}$. Let now the motion, which is ſuppoſed to be actually communicated to the rod at D, be equivalent to the motion of a body whoſe magnitude is G, and moving with a velocity v; then if that motion be communicated at G, the velocity of the center of gravity is well known to be

$= \frac{G \times v}{A+B}$; hence $\frac{B \times BG^2 + A \times AG^2}{A \times B \times AB} : \frac{B \times BD \times BG + A \times AD \times AG}{A \times B \times AB} :: \frac{G \times v}{A+B} : \frac{G \times v}{A+B} \times \frac{B \times BG \times BD + A \times AD \times AG}{B \times BG^2 + A \times AG^2}$ = the velocity of the center of gravity, when the ſame motion is actually communicated to any point D. Now $BD = BG + GD$, and $AD = AG - GD$; hence $B \times BG \times BD + A \times AD \times AG = B \times BG^2 + A \times AG^2 + GD \times \overline{B \times BG - A \times AG}$ = (becauſe $B \times BG - A \times AG = 0$) $B \times BG^2 + A \times AG^2$; conſequently the velocity becomes $\frac{G \times v}{A+B}$; and hence the center of gravity moves with the ſame velocity, wherever the motion is communicated.

PROP. III.

Let a given elaſtic body P, *moving with a given velocity, be ſuppoſed to ſtrike the lever at the point* D *in a direction perpendicular to it; to determine the velocity of the center of gravity* G *after the ſtroke.*

Suppoſe firſt the body to be non-elaſtic, and let v be the velocity of the center of gravity after the ſtroke upon that ſuppoſition, and V the velocity of the ſtriking body: then $CG : CD :: v : \frac{v \times CD}{CG}$ = the velocity of the point D after the ſtroke, or of the body P; for the ſame reaſon $\frac{v \times CA}{CG}$ and $\frac{v \times CB}{CG}$ equal the velocities of A and B reſpectively. Now, becauſe in revolving bodies, the momenta, ariſing from the magnitude of the bodies, their diſtance from the center of rotation and velocity conjointly, remain the ſame after the ſtroke as before, we ſhall have $P \times V \times DC = \frac{v \times CD^2 \times P}{CG} + \frac{v \times CA^2 \times A}{CG} + \frac{v \times CB^2 \times B}{CG}$, and therefore $v = \frac{P \times V \times DC \times CG}{P \times DC^2 + A \times AC^2 + B \times BC^2} = \frac{P \times V \times CG}{\overline{A+B} \times CG + P \times DC}$; hence if P be ſuppoſed an elaſtic body, we ſhall have $\frac{2 \times P \times V \times CG}{\overline{A+B} \times CG + P \times DC}$ for the velocity of the center of gravity after the ſtroke, *in ipſo motûs initio.*

PROP. IV.

To determine the motion of the bodies after the firſt inſtant, or when they are left to move freely by themſelves.

The writers on mechanics, from conſidering the equality of motion on each ſide the center of gravity, when a body revolves about that point, have inferred, that if a body had a projectile as well as a circular motion communicated to it, the center of gravity would continue to move in a right line, as that point would not be diſturbed by the rotatory motion: yet, as, in the caſe we are now conſidering, the bodies begin to revolve about a different center, it may be proper to examine more accurately into this matter, and to ſhew from what principle it is that the motion of the center of gravity is preſerved in a right line.

Let a motion perpendicular to the rod be communicated to A (fig. 2.) and then by Cor. 3. Prop. I. B will not be diſturbed by ſuch an action; and A will in the firſt inſtant have a tendency to revolve about B as a center, and would actually deſcribe the arc AH, if the body B were fixed: let the angle ABH be ſuppoſed infinitely ſmall, and let GK be the arc, the center of gravity would have deſcribed, and draw the tangents AF, G*g* to the arcs AH, G*g* reſpectively. Now, if A could have moved freely, it would (becauſe AF = AH) have deſcribed AF in the ſame time the arc AH was deſcribed, upon ſuppoſition that B was fixed; for the radius BA being perpendicular to the circular arc AH, the force of the lever could have no efficacy to accelerate or retard the motion of A in the arc AH, and therefore the velocity in that arc is the ſame as it would have been if it had moved freely in the tangent: hence HF is that ſpace through which the centrifugal force of A would have carried that body, could it have moved freely; but as A is connected to B by means of the lever, it is manifeſt that the ſame force which would have carried A from H to F in the direction of the lever, will, when it has both bodies to move, carry it over a ſpace which is to FH as A : A + B, or as B*g* : BH, or as *g*K : FH; hence that ſpace, or the ſpace through which the centrifugal force of A will draw the lever in the direction BH, is equal to K*g*; that is, the point K, which is the center of gravity of A and B, will be found at *g*, and conſequently the center of gravity has preſerved its motion uniform in the right line G*g*, inaſmuch as the centrifugal force, acting perpendicularly to the direction of the center of gravity, can neither accelerate or retard its motion. In the ſame manner it may be proved, that the motion of the center of gravity is continued uniform in the ſame right line, whatever be the poſition of the lever. Moreover, as the centrifugal force acts in the direction of the lever, it cannot alter its angular velocity, which will therefore remain as *in ipſo motûs initio*. If now we ſuppoſe that to the force impreſſed upon A, two other equal accelerative forces be communicated to A and B at the ſame time, it is evident that no alteration can ariſe from the actions of the bodies on each other; and the caſe will then be ſimilar to the motion of the bodies, ſuppoſing a ſingle force had been impreſſed at any point D. The like method of reaſoning may be extended to any number of bodies.

The ſame thing may alſo eaſily be demonſtrated in the following manner. The centrifugal forces of A and B (fig. 1.) are reſpectively $A \times AC$ and $B \times BC$; alſo the centrifugal force of the point G, conſidering it as having both bodies to move in the direction of the rod, is $\overline{A+B} \times GC$, but from mechanics $A \times AC + B \times BC = \overline{A+B} \times GC$: hence the centrifugal forces of the bodies A and B give the center of gravity a centrifugal force equivalent to its own centrifugal force, which, as the latter would cauſe that center to move in the tangent G*g*, the lever not being fixed at C, it is manifeſt that the former will cauſe the center of gravity to continue its motion in the ſame direction.

That this motion of the lever, in a direction from the center C, is the only motion which is communicated to it from the effect of the bodies A and B is manifeſt from hence. The bodies begin to revolve freely about the point C, and conſequently if the point C had been fixed, the bodies would have moved on with a uniform angular velocity about C; if therefore we ſuppoſe the lever not to be fixed at C, as the efficacy of the centrifugal force which acts in the direction of the lever is now ſuffered to take place, and no new external force is impreſſed on either of the bodies, it is manifeſt, that if in the former caſe the bodies had no efficacy to diſturb the angular velocity of the lever, they cannot have any in the latter, conſequently the angular velocity, and from what has been before proved, the uniform motion of the center of gravity in a right line, remain unaltered, after the commencement of the motion.

PROP. V.

In the time the bodies make one revolution, the center of gravity will move over a ſpace equal to the circumference of a circle whoſe radius is CG (*fig.* 1.)

From the laſt Propoſition, the angular velocity of the lever is continued uniform; hence the time of a revolution is juſt the ſame as if the point C were fixed, and the bodies were to continue to revolve about that point as a center, in which caſe the center of gravity G, in the time of a revolution, would evidently deſcribe the circumference of a circle whoſe radius is GC. This therefore is the ſpace the center of gravity deſcribes in a right line when the bodies move freely, for from the laſt Propoſition that center is carried uniformly forward with the ſame velocity.

Cor. 1. Hence if the magnitude of the force acting at D vary, the velocity of the center of gravity will vary in the ſame ratio as the angular velocity.

Cor. 2. Hence the point D may be found, where a force being applied, the bodies ſhall make one revolution, whilſt the center of gravity moves over any given ſpace (s): for let $p =$ the periphery of a circle whoſe radius is unity, then $p : 1 :: s : \frac{s}{p} =$ the radius of a circle whoſe circumference is the ſpace to be paſſed over in the time of a revolution, and which muſt therefore, by the Propoſition, be equal to GC; the point C therefore being determined, D may be eaſily found, for from mechanics $CG \times DG$ is given; and from Cor. 3. Prop. I. when D comes to A, C will coincide with B, $\therefore CG \times GD = AG \times GB$, and conſequently $DG = \frac{AG \times GB}{CG}$.

PROP. VI.

To determine the time of one revolution, ſuppoſing every thing given as in Prop. III.

The point D being given, we have from Cor. 2. to the laſt Propoſition, $CG = \frac{AG \times GB}{DG}$; put W equal the circumference of a circle whoſe radius is CG, and it appears from the laſt Propoſition, that W is the ſpace the center of gravity paſſes over in the time of one revolution; hence, becauſe from Prop. IV. the center of gravity moves uniformly, we have by Prop. III. $\frac{2 \times V \times P \times CG}{\overline{A+B} \times CG + P \times DC}$

$: 1''$

$: 1'' :: \text{w} : \text{w} \times \frac{2 \times V \times P \times CG}{\overline{A+B} \times CG + P \times DC}$ = the time of one revolution.

Cor. Hence the angular velocity being inverſely as the time of a revolution, will vary as $\frac{\overline{A+B} \times CG + P \times DC}{V \times P \times CG \times W}$.

P R O P. VII.

The point C, *as the center of gravity moves forward, will deſcribe the common cycloid.*

From the deſcription of the common cycloid it appears, that the center of the generating circle paſſes over a ſpace equal to the circumference of that circle whilſt it makes one revolution. With the center G (fig. 3.) and radius GC, deſcribe the circle C*xy*, and draw CR, GW perpendicular to ABC, and let the circle C*xy* be ſuppoſed to revolve on the line CR; then will the center G move over a ſpace equal to the circumference of the circle C*xy* whilſt it makes one revolution, and the point C will deſcribe the common cycloid: but from Prop. V. the point G *will* move over a ſpace equal to the circumference of a circle whoſe radius is GC, whilſt the bodies, and conſequently GC, make one revolution; and hence the point C will deſcribe the ſame curve as before, that is, the common cycloid.

PROP.

PROP. VIII.

Let a motion be communicated to the lever obliquely, to determine the point about which the bodies begin to revolve.

Let FD (fig. 4.) repreſent the force communicating the motion at the point D, which reſolve into two others FH, HD, the former FH parallel to the lever, and the latter HD perpendicular to it. Let C be the point about which the bodies would have begun to revolve, had the force HD only acted, and which may be found by Prop. I: and ſuppoſe in this caſe *mgn* to have been the next poſition of the lever after the commencement of the motion, or that the bodies A, B, and center of gravity G, had been carried to *m*, *g* and *n* reſpectively. But as the force FH acts at the point D at the ſame time in the direction of the rod, if we take G*q* : G*g* as FH : HD, then whilſt the center of gravity would have moved from G to *g* in conſequence of the force HD, it will by means of the force FH be carried in the direction of the lever from G to *q*, and alſo every other point of the lever will be carried in the ſame direction with the ſame velocity; take therefore A*p* and B*r* each equal to G*q*, and complete the parallelograms A*a*, G*w* and B*b*, and the bodies A, B, and center of gravity G will, at the end of that time, be found at *a*, *b* and *w* reſpectively, and *awb* will be the poſition of the lever. Now it is evident, that C is not the point about which the bodies begin to revolve, for (conſidering the lever to be produced to C) that point muſt have moved over a ſpace C*c* equal to G*q*, when the lever is come into the poſition *awb*: draw CO perpendicular to CB, and GO perpendicular to G*w*, and O will be the center of rotation at the commencement of the motion. For conceive CO to be a lever, then the lever ABC has a circular motion about C, whilſt that point is moving from C to *c*, and conſequently the point O is carried forward in a direction parallel to C*c* by this motion; but as the lever CO is carried by a circular motion about C in a contrary direction, it is evident that that point of the lever CO muſt be at reſt where theſe two motions are equal, as they are in contrary directions. Now the velocity of C in the direction C*c* : velocity of G about C : : G*q* : G*g* : : (by ſim. triang.) CO : CG, and the velocity of the point G about C : velocity of the point O about C : : CG : CO; hence *ex æquo* the velocity of C in the direction of C*c*, or of O in the direction OP parallel to C*c*, is equal to the velocity of the ſame point O in a contrary direction ariſing from its rotation about C, and conſequently O being a point at reſt, muſt be the center of rotation *in ipſo motûs initio*. Alſo, becauſe *ma* is equal and parallel to *nb*, *ab* muſt be equal and parallel to *mn*, therefore the angular velocity is juſt the ſame as if the force FH had not acted. The center O of rotation at the beginning of the motion being thus determined, every thing relative to the motion of the bodies, after they are at liberty to move freely, may be determined as in the preceding Propoſitions.

Cor. 1. Hence it appears, that whatever be the magnitude or direction of the force communicating the motion, or the point at which it acts, the center of gravity will move in a line parallel to the direction of the force, for the triangles FHD, G*qw* being ſimilar, G*w* muſt be parallel to FD.

Cor. 2. The ſame is manifeſtly true for any number of bodies; for let (fig. 5.) E be a third body, and conceive it to be connected with the other two bodies A and B in their center of gravity G; then if FD repreſents the force acting at the point D, it is evident from the laſt Corol. and the ſecond Prop. that the center of gravity moves with the ſame velocity and in the ſame direction, as if the ſame motion had been communicated at G in a line RG parallel to FD, and that the center of gravity has the ſame velocity communicated to it, as if the two bodies had been placed at G; conceive therefore the bodies A and B to be placed at G, and let the force act at D, and then from the laſt Corol. the center of gravity *g*, of the three bodies, will move in a line parallel to the direction of the force communicated. In the ſame manner it may be proved for any number of bodies.

SCHOLIUM.

The method here made uſe of to determine the point of rotation *in ipſo motûs initio*, when a ſingle force acts at any point D, may be applied, when any number of forces act at different points at the ſame time. For let (fig. 1.) α, β, γ, &c. repreſent the forces acting on the lever at the points D, E, F, &c. reſpectively; then from the ſame principles the effect of all the forces on A : the effect on B : : $\frac{\alpha}{AD} + \frac{\beta}{AE} + \frac{\gamma}{AF} + \text{\&c.} : \frac{\alpha}{BD} + \frac{\beta}{BE} + \frac{\gamma}{BF} + \text{\&c.}$ which quantities put equal to P and Q reſpectively, and then $\frac{P}{A} : \frac{Q}{B} : : Am : Bn : : AC : BC$, from whence it appears, that (putting $GC + GA = AC$ and $GC - GB = BC$) the diſtance $GC = \frac{A \times Q \times AG + B \times P \times BG}{B \times P - A \times Q}$. The ſame concluſion might have been deduced from this conſideration; that if any number of forces act on a lever, the effect on any point of that lever is juſt the ſame as if a force, equivalent to the ſum of theſe forces,

had

had acted at their common center of gravity, find therefore their common center of gravity, and conceive a force equivalent to them all to be communicated to that point, and the Problem is reduced to the caſe of the firſt Propoſition. If any of the forces had acted on the oppoſite ſide of the lever, ſuch forces muſt have been conſidered as negative.

If there be any number of bodies placed on the lever, and a ſingle force acts at D, it will appear from the ſame principles that the point C, about which they begin to revolve, will be the point of ſuſpenſion to the center of percuſſion D; and the ſame concluſion will be obtained, if the bodies be not ſituated in a ſtraight line. As a direct inveſtigation, however, is always to be preferred to concluſions drawn from induction, it may be thought proper, before we apply any of the foregoing principles to the caſe of the action of bodies upon each other by impact, to ſhew how ſuch a direct inveſtigation to determine the point about which a body, having a motion communicated to it, begins to revolve, may be obtained; previous to which, however, ſome further conſiderations are neceſſary.

PROP.

PROP. IX.

If a force acts upon a body in any given direction not paſſing through the center of gravity; to determine the plane of rotation, the direction in which the center of gravity begins to move, and its motion after.

Conceive a plane AyBZ (fig. 6.) to be ſupported upon a line AB paſſing through its center of gravity G, and ſuppoſe a force to act at any point D in that line, and in a direction perpendicular to the plane; then it is manifeſt, that ſuch a force can give the plane no rotatory motion about AB. Imagine now the ſupport to be taken away whilſt the force is acting at D, then it is evident, that as the plane had no tendency to move about AB as an axis, and the taking away of the ſupport can give it no ſuch motion, it will, by Cor. 2. Prop. VIII. begin its progreſſive motion in the direction in which the force acts; and as the force is ſuppoſed not to act at the center of gravity, it muſt at the ſame time have a rotatory motion about ſome axis, which, as it has no motion about AB, muſt lie ſomewhere in the plane, and perpendicular to AB; and conſequently *in ipſo motûs initio* the plane of rotation muſt be perpendicular to the plane AyBZ.

A*y*BZ. Let LCM, perpendicular to AB, be the axis about which the plane begins to revolve, and *p*, *q* be two equal particles of the plane ſimilarly ſituated in reſpect to AB, alſo *qb*, *pa* perpendicular to LCM. Now the centrifugal force of *p*, or its force in the direction *ap* is *p* × *ap*, and that of *q* in the direction *bq* is *q* × *bq*; to determine now how theſe forces will affect the motion of the plane, we may obſerve in the firſt place, that the force *p* × *ap*, acting at *a* in the plane, muſt tend to give it a motion about an axis perpendicular to the plane; but as an equal force *q* × *qb* acts at *q* to give it a motion in a contrary direction, it is evident that the two forces will deſtroy each other, ſo far as they tend to generate any motion in the plane about an axis perpendicular to it; and hence it is manifeſt, that if the parts of the plane A*y*B, AZB, be ſimilar, and ſimilarly ſituated in reſpect to AB, the plane, after the commencement of the motion, will have no tendency to revolve about an axis perpendicular to it. Alſo, as the centrifugal force of each particle acts in a direction parallel to AB, it can give the plane no tendency to revolve about that line as an axis, and conſequently the plane of rotation will be preſerved as *in ipſo motûs initio*. Conceiving therefore the plane on each ſide the line AB to be ſimilar, and ſimilarly ſituated, ſuppoſe another plane to be fixed upon this, whoſe parts on each ſide AB are ſimilar, and ſimilarly ſituated, and the force to act as before, then it is manifeſt, that as each plane endeavours to preſerve the ſame plane of rotation, the two planes connected will alſo continue to move in the ſame plane of rotation, for the action of one plane on another, on each ſide the plane of rotation, being equal, cannot tend to diſturb the motion in that plane; and as this muſt be true for any number of planes thus ſimilar and ſimilarly ſituated, it is evident, that if a force ſhould act upon a body, and each ſection, perpendicular to the direction of the force, ſhould be ſimilar on each ſide the plane paſſing through the direction of the force, and the center of gravity of the body, that that plane would be the plane of rotation in which the body would both begin and continue its motion. It appears alſo from what has been proved, that if every ſection on each ſide that plane had not been ſimilar, the plane of rotation would not *neceſſarily* have continued the ſame after the commencement of the motion. Hence all bodies, formed by the revolution of any plane figure, will have the axis about which they were generated, a fixt axis of rotation; to determine, however, every other axis of a body about which it would continue to revolve, would be foreign to the ſubject of this paper. Suppoſing therefore the plane of rotation to continue the ſame (for in this paper I mean to confine my enquiries to ſuch caſes) imagine all the particles of the body to be referred to that plane orthographically, which ſuppoſition not affecting the angular motion of the body, the centrifugal force of all the particles, to cauſe the body to revolve about an axis perpendicular to that plane, will remain unaltered. Let LMNO (fig. 7.) be that plane, and ſuppoſe a force to act at A in the direction PA lying in the ſame plane, which produce until it meets LN, paſſing through the center of gravity G, perpendicularly in D; then by Cor. 2. Prop. VIII. the center of gravity G will begin its motion in a line parallel to PA, or perpendicular to LN; and conſequently the center C, about which the body begins to revolve, muſt lie ſomewhere in the line LN. Now the centrifugal force of any particle p is $p \times pc$; let fall pa perpendicular to LN, then the effect of that force at C, in a direction perpendicular to LN, will be $p \times pa$, and in the direction CL it will be $p \times \mathrm{C}a$; but as the ſum of all the quantities $p \times pa = 0$, and the ſum of all the quanties $p \times \mathrm{C}a =$ the body multiplied into CG, it follows from the ſame reaſoning as in Prop. III. that the point G will continue to move in a direction perpendicular to LN; and alſo, as the forces $p \times \mathrm{C}a$ act in a direction perpendicular to that in which the center of gravity moves, its motion muſt be continued uniform. In the following Propoſitions, therefore, we ſuppoſe the axis of the body, after the commencement of the motion, to continue perpendicular to the plane paſſing through the direction of the force, and the center of gravity of the body, and that the body itſelf is orthographically projected upon that plane; alſo in the caſe of the action of two bodies on each other, the plane paſſing through the direction of the ſtriking body and point of percuſſion is ſuppoſed to paſs through the centers of gravity of each body; that the axis of each body after it is ſtruck continues perpendicular to that plane, and that each body is reduced to it in the manner above deſcribed.

PROP. X.

To determine the point about which a body, when ſtruck, begins to revolve.

Let LMNO (fig. 7.) repreſent the body, G the center of gravity, and PA the direction of the force acting at A, which produce till it meets LN, paſſing through G, perpendicularly in the point D; draw *pb* perpendicular to *p*C, on which (produced if neceſſary) let fall the perpendicular D*w*; C being ſuppoſed the point about which the body begins to revolve, and which, from the laſt Propoſition, is ſomewhere in the line LN. Becauſe the body, in conſequence of the force acting at D, begins to revolve about C, and conſequently if immediately after the beginning of the motion a force were applied at D equal to it, and in a contrary direction, the motion of the body would be deſtroyed, it is evident, that the efficacy of the body revolving about C, to turn the body about D, ſhould any obſtacle be oppoſed to its motion at that point, muſt be equal to nothing; for were it not, the body, when ſtopped at D, would ſtill have a rotatory motion about that point, and conſequently two equal and oppoſite forces applied at D would not deſtroy each others effects, which would be abſurd. Now the force of a particle p, in the direction pw, being $p \times pC$, its efficacy to turn the body about the point D is $p \times pC \times Dw$; but by ſim. triang. $Dw : Db :: aC : pC$, $\therefore Dw = \frac{Db \times aC}{pC}$, and conſequently the efficacy to turn the body about $D = p \times Db \times aC = p \times Ca \times \overline{DC - Cb} = p \times Ca \times DC - p \times pC^2$; hence the ſum of all the $p \times Ca \times DC$ – the ſum of all the $p \times PC^2 = 0$, and conſequently $CD = \frac{\text{ſum of all the } p \times PC^2}{\text{ſum of all the } p \times Ca}$, therefore D is the center of percuſſion, the point of ſuſpenſion being at C.

Cor. From this and the preceding Propoſition it appears, that every thing which was proved in Prop. V. VI. VII. holds here alſo in the caſe of the action of one body on another.

PROP.

PROP. XI.

Let a body P (fig. 8.) *moving with the velocity* V, *ſtrike the body* Q *at reſt in the point* A, *and in a direction* AD *paſſing through the center of gravity of the ſtriking body; to determine the velocity of each body after the ſtroke, ſuppoſing them to be elaſtic.*

The ſolution of this Propoſition depending on the ſame principles as that of Prop. III. we ſhall have, putting v equal the velocity of the center of gravity G after the ſtroke, on ſuppoſition that the bodies were non-elaſtic (DGC being ſuppoſed perpendicular to AD, and C the point about which the body Q begins to revolve) $V \times P \times CD = \frac{v \times P \times CD^2}{CG} + \frac{v \times \text{ſum of all the } p \times Cp^2}{CG}$, and conſequently $v = \frac{V \times P \times CD \times CG}{\text{ſum of all the } p \times pC^2 + P \times CD^2}$; but it is well known, that the ſum of all the $p \times pc^2 = CG \times CD \times Q$, and hence $v = \frac{V \times P \times CG}{Q \times CG + P \times DC}$, and therefore if the bodies be ſuppoſed elaſtic, we have $\frac{2P \times V \times CG}{Q \times CG + P \times DC}$ for the velocity of the center of gravity G after the ſtroke. Now to determine the velocity of P, we have $\frac{P \times V \times CD}{Q \times CG + P \times DC}$ equal its velocity after the ſtroke from ſingle impact, and conſequently $V - \frac{P \times V \times CD}{Q \times CG + P \times DC} = \frac{Q \times V \times CG}{Q \times GC + P \times DC}$ is the velocity loſt by P from ſimple impact; hence if the bodies be elaſtic, $\frac{2\times Q\times V\times CG}{Q\times GC+P\times DC}$ will be the velocity loſt by P if elaſtic, and conſequently the velocity of P after the ſtroke $= V - \frac{2\times Q\times V\times CG}{Q\times GC+P\times DC} = \frac{P\times DC - Q\times GC}{Q\times GC+P\times DC}\times V$.

Cor. 1. If the direction AD paſſes through G, then CG being equal to CD, we have $\frac{2PV}{Q+P}$ = Q's velocity, and $\frac{P-Q}{P+Q}\times V$ = P's velocity, which is well known from the common principles of elaſtic bodies.

Cor. 2. If $P\times DC = Q\times GC$, or $P : Q :: GC : DC$, then will the body P be at reſt after the ſtroke.

Cor. 3. If Q were infinitely great, the velocity of P after the ſtroke would be $= -V$ as it ought, for P would then ſtrike againſt an immoveable obſtacle.

Cor. 4. Whatever motion Q gains from the action of P, it would loſe, if, inſtead of ſuppoſing P to ſtrike Q, Q were to move in an oppoſite direction, and ſtrike P at reſt with the ſame velocity with which P ſtruck Q; in ſuch caſe, therefore, the velocity of Q after the ſtroke would be $V - \frac{2P\times GC\times V}{Q\times GC+P\times DC} = \frac{\overline{Q-2P}\times GC+P\times DC}{Q\times GC+P\times DC}\times V$.

Cor. 5. Hence if P be infinitely great, or Q be ſuppoſed to ſtrike an immoveable object, its velocity after the ſtroke will be $= \frac{DC-2CG}{DC}\times V$: hence when $DC = 2GC$, the body Q will have no progreſſive motion after the ſtroke, but would in ſuch caſe, if P were immediately taken away, continue to revolve about a fixed axis. It may alſo be obſerved, that when DC is greater than 2GC, or the velocity of Q is poſitive, that, becauſe it is impoſſible for Q to continue its progreſſive motion, it is only to be underſtood, that if immediately after the impact the body P were removed, the body Q would then proceed with ſuch a velocity.

Cor. 6. Suppoſe the bodies to be non-elaſtic, and let M be the magnitude of a body placed at D, which, being acted upon by P, ſhall have the ſame velocity generated as was before generated in the point D of the body Q; then by the common rule for non-elaſtic bodies, the velocity of M after the ſtroke will be $\frac{P\times V}{P+M}$, and hence $\frac{P\times V}{P+M} = \frac{P\times V\times DC}{Q\times CG+P\times DC}$, conſequently $M = Q\times\frac{GC}{DC}$.

Cor. 7. If a given quantity of motion were communicated to *any* point of the body Q, the progreſſive motion of that body after the ſtroke would be the ſame. For ſuppoſe the magnitude of the body P to be diminiſhed *ſine limite*, and its velocity to be increaſed in the ſame ratio, then, becauſe $\frac{P\times V\times CD}{Q\times CG+P\times DC}$ (which is the velocity of P after the ſtroke, if the bodies be non-elaſtic) = (becauſe P is infinitely ſmall) $\frac{P\times V\times CD}{Q\times CG}$, the velocity of P after the ſtroke from ſimple impact is finite, conſequently its motion muſt be infinitely ſmall, and therefore P muſt have communicated all its motion to Q: now in this caſe the velocity of Q $(= \frac{P \times V \times CG}{Q \times CG + P \times CD}) = \frac{P \times V}{Q}$, which quantity is independent of the place where the force acts; in the ſame manner it would appear if we had ſuppoſed the bodies elaſtic.

PROP. XII.

Suppoſing every thing given as in the laſt Propoſition, except that the direction AD *does not paſs through the center of gravity g of the ſtriking body; to determine the velocity of each body after the ſtroke.*

Let AD (fig. 9.) be produced to meet F*g*O paſſing through *g*, the center of gravity of the ſtriking body, perpendicularly in F, and ſuppoſe O to be the point of the body P which is not diſturbed by the action of P on Q: now it appears from Cor. 6. Prop. XI. that if both bodies were non-elaſtic, and a body equal to $Q \times \frac{CG}{CD}$ were placed at D, the velocity of that body, from the action of P, would be equal to the velocity of the point D of the body Q; for the ſame reaſon, therefore, it appears, that if, inſtead of ſuppoſing

P to ſtrike Q in the direction FA, a body equal to $P \times \frac{GO}{FO}$ were to ſtrike Q at the ſame point and in the ſame direction (which direction is ſuppoſed to paſs through the center of gravity of that body) the effect on Q would be the ſame; hence, if in the quantity $\frac{V \times P \times CD}{Q \times GC + P \times DC}$, which from the laſt Prop. expreſſes the velocity of the point D after the ſtroke, on ſuppoſition that the bodies are non-elaſtic, we ſubſtitute for P a body equal to $P \times \frac{GO}{FO}$, we ſhall have $\frac{V \times P \times DC \times gO}{Q \times GC \times FO + P \times gO \times DC}$ for the velocity of the point D from the action of P; and conſequently $\frac{2 \times V \times P \times GC \times gO}{Q \times GC \times FO + P \times gO \times DC}$ = the velocity of the center of gravity G of the body Q, after the ſtroke, if the bodies be perfectly elaſtic. To determine now the velocity of the ſtriking body, let of, perpendicular to og, be the ſpace deſcribed by the point O in the firſt inſtant of time after the ſtroke, which, as that point is not diſturbed by the action of the bodies on each other, may repreſent the velocity of P before the ſtroke, and let Fb repreſent the velocity of the point F after the ſtroke; join fb, and draw gd perpendicular to og, and then will gd repreſent the velocity of the center of gravity g of the ſtriking body after the ſtroke. Draw fc perpendicular to FA, and produce gd to meet fc in e; now the velocity loſt by P at the point F by ſimple impact being equal to $V - \frac{V \times P \times DC \times gO}{Q \times GC \times FO + P \times gO \times DC} = \frac{V \times Q \times GC \times FO}{Q \times GC \times FO + P \times gO \times DC}$, we ſhall have *bc* the velocity loſt by the point F, on ſuppoſition that the bodies are perfectly elaſtic (ſuppoſing *of* to repreſent the value of V) equal to $\frac{2 \times V \times Q \times GC \times FO}{Q \times GC \times FO + P \times gO \times DC}$, and therefore by ſim. triang. *fc* (FO) : *cb* : : *fe* (O*g*) : *ed* = $\frac{2 \times V \times Q \times GC \times gO}{Q \times GC \times FO + P \times gO \times DC}$ = the velocity loſt by the center of gravity *g*, and hence $V - \frac{2 \times V \times Q \times GC \times gO}{Q \times GC \times FO + P \times gO \times DC} = \frac{V \times Q \times GC \times FO + V \times P \times gO \times DC - 2 \times V \times Q \times GC \times gO}{Q \times GC \times FO + P \times gO \times DC}$ = the velocity of P after the ſtroke. Now, as it appears from Prop. IX. that the progreſſive motion of a body, when left to move freely, continues uniform and in the ſame direction, it follows, that the expreſſions for the velocities of each body in the firſt inſtant after the ſtroke, both in this and the preceding Propoſition, will repreſent the uniform progreſſive velocities with which the bodies will continue to move, and conſequently the place of each body, at the end of any given time after impact, may eaſily be determined.

Cor. I. If the direction FA paſſes through *g*, then FO and *g*O becoming infinite, we ſhall have $\frac{2 \times V \times P \times GC}{Q \times GC + P \times DC}$ for the velocity of Q, and $\frac{V \times P \times DC - V \times Q \times GC}{Q \times GC + P \times DC}$ for the velocity of P, agreeable to what was proved in the laſt Propoſition.

Cor. 2. Hence the point about which P begins its rotatory motion may eaſily be found; for produce (if neceſſary) *fb* and OF to meet in *a*, and *a* will be the point required; and by ſim. triang. $bc\ (=\frac{2\times\mathrm{V}\times\mathrm{Q}\times\mathrm{GC}\times\mathrm{FO}}{\mathrm{Q}\times\mathrm{GC}\times\mathrm{FO}+\mathrm{P}\times g\mathrm{O}\times\mathrm{DC}}) : cf :: f\mathrm{o}\ (=\mathrm{V}) : \mathrm{o}a = \frac{\mathrm{Q}\times\mathrm{GC}\times\mathrm{FO}+\mathrm{P}\times g\mathrm{O}\times\mathrm{DC}}{2\times\mathrm{Q}\times\mathrm{GC}}$, and hence $\mathrm{F}a=\frac{\mathrm{P}\times g\mathrm{O}\times\mathrm{DC}-\mathrm{Q}\times\mathrm{GC}\times\mathrm{OF}}{2\times\mathrm{Q}\times\mathrm{GC}}$.

Cor. 3. If, inſtead of ſuppoſing Q to have been at reſt, it had been moving forward in a direction parallel to that of the body P, with the velocity v, the motion of each body after the ſtroke may eaſily be determined: for conſidering P as acting upon Q with the velocity $\mathrm{V}-v$, we ſhall have by this Propoſition (putting $2\mathrm{M}=\frac{2\mathrm{P}\times\mathrm{GC}\times\mathrm{GO}}{\mathrm{Q}\times\mathrm{GC}\times\mathrm{FO}+\mathrm{P}\times g\mathrm{O}\times\mathrm{DC}}$) $\overline{\mathrm{V}-v}\times 2\mathrm{M}=$ the velocity communicated to G, therefore $v+\overline{\mathrm{V}-v}\times 2\mathrm{M}=$ the velocity of Q after the ſtroke: alſo $\overline{\mathrm{V}-v}\times\mathrm{M}\times\frac{\mathrm{CD}}{\mathrm{CG}}=$ the velocity gained by the point D from ſimple impact, and conſequently the velocity of that point after $=v+\overline{\mathrm{V}-v}\times\mathrm{M}\times\frac{\mathrm{CD}}{\mathrm{CG}}$, hence $\mathrm{V}-v-\overline{\mathrm{V}-v}\times\mathrm{M}\times\frac{\mathrm{CD}}{\mathrm{CG}}=$ the velocity loſt by P at the point F from ſimple impact, therefore P's velocity after the ſtroke $=\mathrm{V}-\overline{\mathrm{V}-v-\overline{\mathrm{V}-v}\times\mathrm{M}\times\frac{\mathrm{CD}}{\mathrm{CG}}}\times\frac{2g\mathrm{O}}{\mathrm{FO}}$. In the ſame manner it might have been determined, had Q moved in an oppoſite direction.

Cor. 4. Hence alſo we may eaſily determine the motion of each body after the ſtroke, ſuppoſing Q had not been moving in a direction parallel to the motion of P, by reſolving Q's motion into two parts, one parallel to the motion of P, and the other perpendicular; and finding by the preceding what would be the effect of the parallel motions, and then compounding Q's motion, after the ſtroke from that conſideration, with the motion it had in a direction perpendicular thereto before the ſtroke.

Cor. 5. The point *a* of the body P will deſcribe (when that body after the ſtroke has any progreſſive motion) the common cycloid.

Cor. 6. Hence, therefore, the times of the revolutions of each body may be determined as in Prop. VI.

Cor. 7. If the bodies had any rotatory motion before impact, every thing relative to the motion of the bodies after the ſtroke might have been determined from the ſame principles.

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Fig. 1.

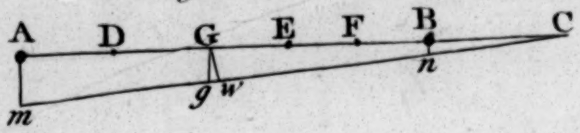

Fig. 2

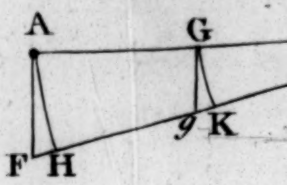

Fig. 4.

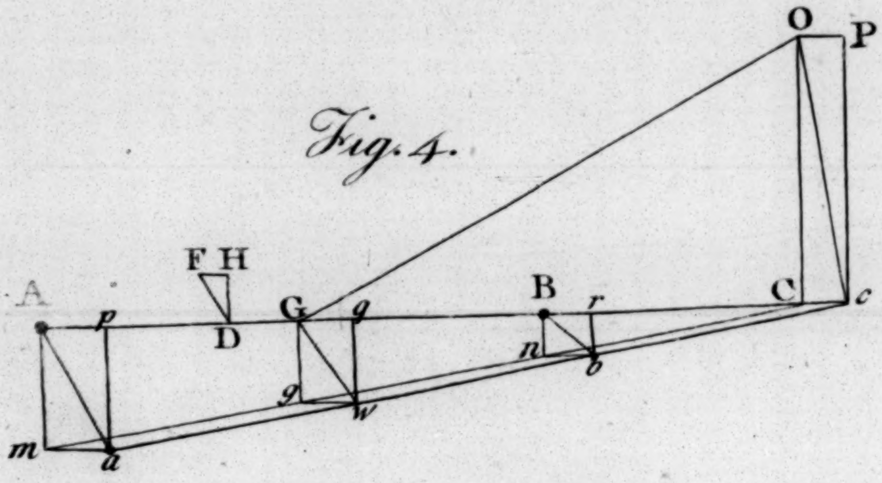

Fig. 7.

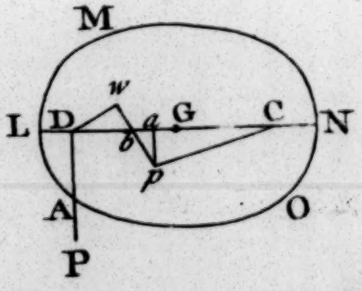

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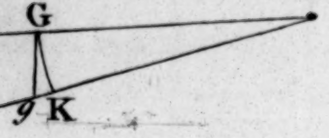
Fig. 2.



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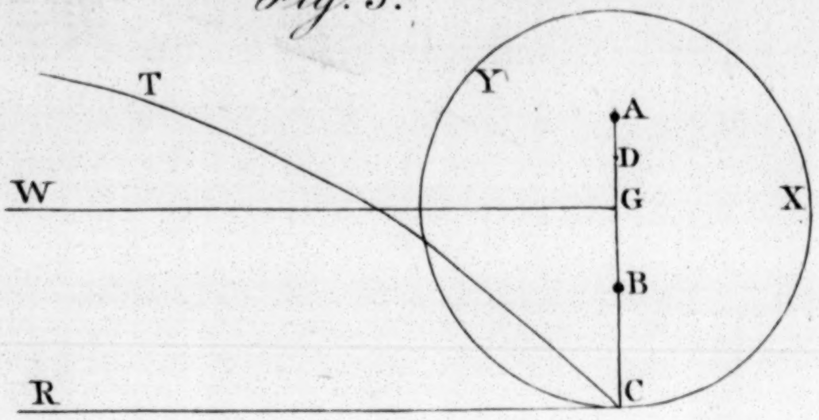


Fig. 6.

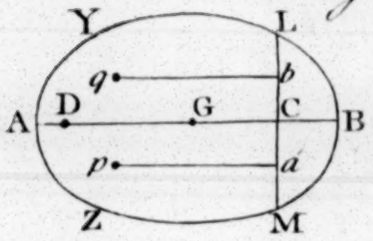
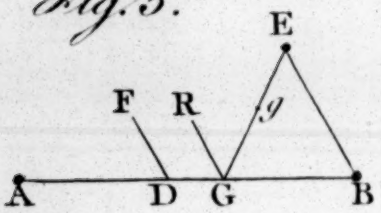


Fig. 5.



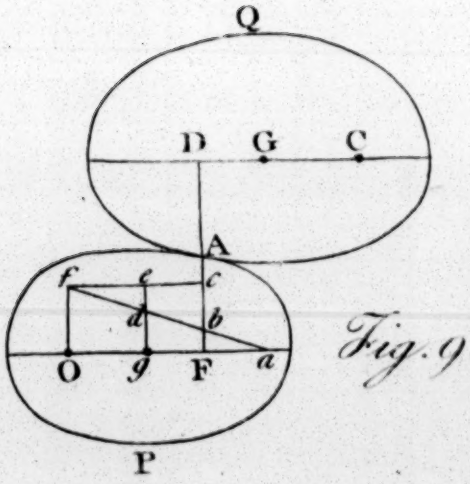
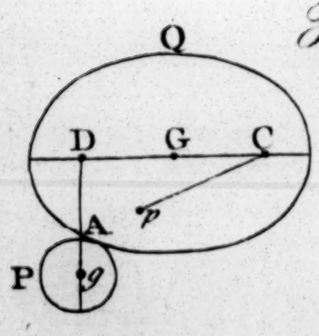


Fig. 9.

Fig. 8.



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